Localizing Limit Cycles, Separatices and Homoclinic Structures

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Abstract

A general method for estimation of domains with limit cycles and finding surfaces with traces of all cycles is proposed. Corresponding estimations of domains with cycles for linear-quadric systems, Van der Pole equations and equations for auto-oscillations in spin generator are indicated.

1 Introduction

The qualitative analysis of nonlinear system of differential equations

$$\dot{x} = f(x), \quad x \in \mathbf{R}^n, \quad f(x) \in C(\mathbf{R}^n) \tag{1}$$

requires to know of such objects as limit cycles, separatrixes and homoclinic structures. It is well-known that they determine the basis properties of systems and the behavior of their solutions. In the further consideration we shall only consider the systems such that there exists a unique solution of (1) for given initial condition.

Any T-periodic solution $x(t)$, $x(t+T) = x(t)$, of nonlinear system (1) specifies the mapping $\mathbf{R} \to \mathbf{R}^n$, $t \to x(t)$, and its image is called a limit cycle (or a cycle) of system (1). In present time there is no analytical method for finding limit cycles of autonomous dynamic systems. Usually limit cycles are sought by means of numerical methods. For using these methods, we need to know two sets. They are a set of initial states and a set containing all limit cycles.

In the present paper we consider localization problem of the limit cycles for dynamic systems (1) in the following settings:

1. For system (1) find a surface S such that any limit cycle of system (1) either intersects or touches S.
2. For system (1) find a set Ω such that all limit cycles of system (1) are in Ω .

In section 2 we give an approach [1-3] to the solution of the above localization problems and develop it for separatrixes and homoclinic structures [4]. In section 3 we demonstrate the efficiency of the method on the example of dynamical system describing auto-oscillations of spin generator. Further, in section 4 we also expound the method for excluding some of the equilibrium points with their neighborhoods from the localization set of cycles. This method is illustrated on the example of the Van der Pole equation. And finally, in section 5 we give some interesting results obtained for two-dimensional systems with linear-quadric right-hand sides.

2 Localization Method

For $\varphi \in C^1(\mathbf{R}^n)$ denote by $L_f\varphi$ derivative of the function φ with respect to system (1), i.e.

$$L_f\varphi(x) = \sum_{i=1}^{n} f_i(x)\partial\varphi(x)/\partial x_i,$$

where $(f_1(x), \ldots, f_n(x))^T = f(x)$.

Theorem 2.1 *For any function $\varphi \in C^1(\mathbf{R}^n)$, any cycle of system (1) contains at least two points of the set [1]*

$$S_\varphi = \{x : \, L_f\varphi(x) = 0\,\}. \tag{2}$$

For the function $\varphi \in C^1(\mathbf{R}^n)$ we put

$$\begin{aligned} \varphi_{sup} &= \sup\{\varphi(x) : \, x \in S_\varphi\,\}, \\ \varphi_{inf} &= \inf\{\varphi(x) : \, x \in S_\varphi\,\}. \end{aligned} \tag{3}$$

Theorem 2.2 *For any function $\varphi \in C^1(\mathbf{R}^n)$, all limit cycles of system (1) belong to the set [1-2]*

$$\Omega_\varphi = \{x : \, \varphi_{inf} \le \varphi(x) \le \varphi_{sup}\}. \tag{4}$$

Collorary 2.1 *All limit cycles of system (1) belong to the set*

$$\Omega = \{\cap\Omega_\varphi, \varphi \in C^1(\mathbf{R}^n)\}. \qquad (5)$$

Actually, it appears that sets (4) contain also all separatrixes, with the ends at finite points, and all homoclinic structures of system (1) [4].

Theorem 2.3 *For any function $\varphi \in C^1(\mathbf{R}^n)$, all separatrixes and homoclinic structures of system (1) belong to the set (4).*

The efficiency of the above results while studying particular systems depends mainly on the proper choice of function φ. That also considerably depends on the form of the set S_φ given by (2). If $S_\varphi = \emptyset$ for some function φ, system (1) has no cycles. The case in which the set S_φ is finite is also simple: one must clarify only the behavior of trajectories that pass through these points.

The functions φ for which the set S_φ is compact are useful. In this case, first, the set (2) is bounded, and this is important for finding periodic solutions by numerical methods, and, secondly, all cycles lie between the level surfaces of the function φ. Surely, it is always worth finding the function φ for which the set (4) is bounded.

3 The Equations for Spin Generator

Among the variety of different radiophysical devices there exist so-called spin generators. If constant and variable magnetic fields are present in the generator then the system of nuclear spins is characterized with common magnetization vector. Following some physical assumptions, the system of nuclear spins can be represented as an oscillating circuit with nuclear resonance. To keep precession of the magnetization vector the circuit is closed with a feedback which is chosen as simple as possible. Supposing an amplifier is in the feedback circuit, then we receive the following system of equations that describes the auto-oscillations in the spin generator [6]:

$$\begin{cases} \dot{x} &= \omega y - \delta_2 x, \\ \dot{y} &= \gamma k z y - \omega x - \delta_2 y, \\ \dot{z} &= c - \delta_1 z - \gamma k y^2, \end{cases} \qquad (6)$$

where ω, γ, k, c, δ_1 and δ_2 are positive parameters.

Theorem 3.1 *For system (6) there exists the two-degree polynomial*

$$\begin{aligned} \varphi(\bar{x}) &= \alpha x^2 + \beta y^2 + \tau z^2 + 2\lambda xy + 2\mu xz + \\ &+ 2\delta yz + 2\theta x + 2\epsilon y + 2\nu z, \end{aligned}$$

where $\bar{x} = (x, y, z)^T$ such that

1^o. $\deg L_f\varphi(\bar{x}) = 2$;

2^o. second-order terms of the function $L_f\varphi(\bar{x})$ form a negative definite quadric form;

3^o. second-order terms of the function $\varphi(\bar{x})$ form a positive definite quadric form.

If the conditions $1^o - 3^o$ are satisfied, the set Ω_φ for system (6) is bounded with an ellipsoid.

If we put $\lambda = \mu = \delta = \theta = \epsilon = \nu = 0$, $\alpha = 1$, $\tau = \gamma k \alpha/\delta_1$ and $\beta = \delta_1\tau/(\gamma k)$, we get the function $\varphi(\bar{x}) = x^2 + y^2 + \gamma k z^2/\delta_1$. Then $L_f\varphi(\bar{x}) = -\delta_2 x^2 - \delta_2 y^2 - \gamma^2 k^2 z^2/\delta_1 + c\gamma k z/\delta_1$ and the surface S_φ is the ellipsoid

$$x^2 + y^2 + \frac{\gamma^2 k^2}{\delta_1\delta_2}\left(z - \frac{c}{2\gamma k}\right)^2 = \frac{c^2}{4\delta_1\delta_2} \qquad (7)$$

of the center $O_1(0, 0, c/(2\gamma k))$. It can be easily checked that $\varphi_{\inf} = 0$ and $\varphi_{\sup} = \varrho^2$, where

$$\varrho^2 = \begin{cases} \dfrac{c^2}{\delta_1\gamma k}, & \text{if } \delta_2 = \gamma k \text{ or } \delta_2 \geq \gamma k/2, \\ \max\left\{\dfrac{c^2}{\delta_1\gamma k}, \dfrac{c^2\gamma k}{4\delta_1\delta_2(\gamma k - \delta_2)}\right\}, & \text{otherwise.} \end{cases}$$

It is clear that the condition $\varphi_{\inf} \leq \varphi(\bar{x})$ holds for all $\bar{x}$ and implies no restrictions. On the other hand, the condition $\varphi(\bar{x}) \leq \varphi_{\sup}$ defines the set Ω_φ bounded with the ellipsoid

$$x^2 + y^2 + \frac{\gamma k}{\delta_1} z^2 = \varrho^2 \qquad (8)$$

of center $O_2(0, 0, 0)$.

All cycles of system (6) twice intersect the ellipsoid (7) and belong to the set Ω_φ bounded with the ellipsoid (8). For example, if we put $\gamma = 2.67 \cdot 10^4$, $c = 10^{-6}$, $\delta_1 = 0.1$, $\delta_2 = 0.01$ and $k = 10^6$, we shall have

$$\begin{aligned} S_\varphi = \{\bar{x} \mid 0.01x^2 + 0.01y^2 + \\ + 7.1289 \cdot 10^{15} z^2 - 0.267z = 0\}, \\ \Omega_\varphi = \{\bar{x} \mid x^2 + y^2 + \\ + 2.67 \cdot 10^5 z^2 \leq 2.5 \cdot 10^{-16}\}. \end{aligned}$$

4 Excluding Some Equilibrium Points

We should notice the following fact. If x^0 is an equilibrium point of system (1), i.e. $f(x^0) = 0$, then $L_f\varphi(x^0) = 0$ for any function φ. Hence, $x^0 \in S_\varphi$ and the equilibrium point x^0 belongs to both the set Ω_φ and set Ω (5). But it is possible to exclude some of the equilibrium points together with some their neighborhoods from the localization set of limit cycles. We can suggest the following method for excluding asymptotically stable (in direct or inverse time) equilibrium points of system (1).

Let's choose function $\varphi \in C^1(\mathbf{R}^n)$ such that it satisfies the following conditions:

1^o) $\varphi(x) > 0$ for $x \neq x^0$ and $\varphi(x^0) = 0$;

2^o) in some neighborhood U of x^0 $L_f\varphi(x) < 0$ for $x \neq x^0$;

3^o) outside of U $\varphi(x) > \delta > 0$.

Then, in this neighborhood U the point x^0 is an isolated solution of the equation $L_f\varphi(x) = 0$. Since any limit cycle of system (1) is not able to pass through the equilibrium point x^0, we may exclude it from the set S_φ while finding the values (3). Having done that, let's find $\varphi_{\inf}$ in accordance with (3) and the result will be that $\varphi_{\inf} > 0$. Therefore, set (4) doesn't contain the equilibrium point x^0 with its neighborhood $\{x \in \mathbf{R}^n | \varphi(x) < \varphi_{\inf}\}$. Such a function $\varphi(x)$ is easily built when roots of characteristic equation for linear part of system (1) in the equilibrium point x^0 are of the same sign. In this case as the function φ it is possible to take a Liapunov function for this linear part of system (1). We can choose $\varphi(x)$ as a positively definite quadric form with respect to the variables $x - x^0$.

Example. Consider the Van der Pole equation $\ddot{x} + \epsilon\dot{x}(x^2 - 1) + x = 0$, where ϵ is a positive parameter, and rewrite the equation as the system

$$\dot{x} = y, \qquad \dot{y} = \epsilon y(1 - x^2) - x. \tag{9}$$

System (9) has the equilibrium point $O(0,0)$. If $0 < \epsilon < 2$, then the point O is an unstable focus and if $\epsilon > 2$, O is an unstable knot. Hence, the point O is asymptotically stable in inverse time. Now we will try to find such a localization domain for limit cycles of system (9) that doesn't contain this equilibrium point with its some neighborhood. First, finding linearization of system (9) in some neighborhood of the point O, we obviously obtain

$$\dot{x} = y, \qquad \dot{y} = \epsilon y - x. \tag{10}$$

Since the point $O(0,0)$ is asymptotically stable in inverse time, for a positively definite quadric form $w(x,y) = (x,y)^T W(x,y) > 0$ there exists a positively definite Liapunov function $V(x,y) = v_1x^2 + v_2y^2 + 2v_3xy$, such that its derivation with respect to system (10)

$$\dot{V}(x,y)|_{(10)} = -2v_3x^2 + 2(v_3 + \epsilon v_2)y^2 + \\ + 2(v_1 - v_2 + \epsilon v_3)xy$$

is equal to $w(x,y)$. For

$$W = \begin{pmatrix} 1 & w_1 \\ w_1 & w_2 \end{pmatrix} > 0,$$

i.e. $w_2 - w_1^2 > 0$, we get $v_1 = w_1 + 0.5w_2/\epsilon + 0.5(\epsilon + 1/\epsilon)$, $v_2 = 0.5w_2/\epsilon + 0.51/\epsilon$, $v_3 = -0.5$. Suppose $\varphi = V(x,y)$, then $L_f\varphi(x,y) = \dot{V}(x,y)|_{(9)}$ and thus we have

$$L_f\varphi = \epsilon x^3y - (w_2 + 1)x^2y^2 + x^2 + w_2y^2 + 2w_1xy.$$

The point O is an isolated solution of the equation $L_f\varphi(x,y) = 0$. Since any limit cycle of system (9) is not able to pass through the equilibrium point, we may exclude it from the set S_φ. Consequently, we obtain S_φ given as

$$S_\varphi = \{L_f\varphi = 0\} \setminus \{(0,0)\}.$$

Further, solving the extremum problem for the function φ on the set S_φ, we obtain $\varphi_{\inf} > 0$ and then the domain $\Omega_\varphi = \{(x,y) \mid \varphi(x,y) \geq \varphi_{\inf}\}$. Thus, we succeed to exclude a domain, that doesn't contain any points of any cycles.

For $\epsilon = 2$, $w_1 = 0$ and $w_2 = 0.5$ we obtained

$$\begin{aligned} \varphi(x,y) &= 1.375x^2 + 0.375y^2 - xy, \\ S_\varphi &= \{2x^3y + (1 - 1.5y^2)x^2 + 0.5y^2 = 0\}, \\ \Omega_\varphi &= \{1.375x^2 + 0.375y^2 - xy \geq 1.1463\}. \end{aligned}$$

It should be pointed out that the best results (in a sense of maximum area of the excluded set) are reached if $w_2 - w_1^2 \to 0$.

5 Two-dimensional Linear-Quadric Systems.

Consider the general case of two-dimensional linear-quadric right-hand side dynamic systems given as

$$\begin{cases} \dot{x} = \alpha_0 + \alpha_1x + \alpha_2y + \alpha_3x^2 + \alpha_4xy + \alpha_5y^2, \\ \dot{y} = \beta_0 + \beta_1x + \beta_2y + \beta_3x^2 + \beta_4xy + \beta_5y^2, \end{cases} \tag{11}$$

where $\alpha_i, \beta_i, i = \overline{0,5}$, are the system's parameters. By $\bar{x}$ denote $(x,y)^T \in \mathbf{R}^2$. Let's take

$$\varphi(\bar{x}) = ax + by, \qquad a^2 + b^2 \neq 0. \tag{12}$$

Then

$$L_f\varphi(\bar{x}) = \bar{x}^T B\bar{x} + C\bar{x} + D,$$

where

$$B = \begin{pmatrix} a\alpha_3 + b\beta_3 & (a\alpha_4 + b\beta_4)/2 \\ (a\alpha_4 + b\beta_4)/2 & a\alpha_5 + b\beta_5 \end{pmatrix},$$

$$C = \begin{pmatrix} a\alpha_1 + b\beta_1 \\ a\alpha_2 + b\beta_2 \end{pmatrix},$$

$$D = a\alpha_0 + b\beta_0.$$

The set $S_\varphi = \{\bar{x} | L_f\varphi(\bar{x}) = 0\}$ is a curve of second order and its quadric form defines the curve's type. Denote by $\delta(a,b)$ the determinant of B, i.e.

$$\begin{aligned}\delta(a,b) &= detB = a^2(\alpha_3\alpha_5 - {\alpha_4}^2/4) + \\ &+ b^2(\beta_3\beta_5 - {\beta_4}^2/4) + \\ &+ ab(\alpha_3\beta_5 + \beta_3\alpha_5 - \alpha_4\beta_4/2).\end{aligned}$$

It is obvious that curve $\{L_f\varphi(\bar{x}) = 0\}$ is an ellipse if and only if

$$\delta(a,b) > 0.$$

Definition 5.1 *Say that system (11) admits an elliptic universal section if* $\exists a, b : \delta(a,b) > 0$.

Theorem 5.1 *System (11) hasn't got any elliptic universal section if and only if the quadric form of $\delta(a,b)$ is negative-definite or nonpositive one.*

Theorem 5.2 *If the quadric form of $\delta(a,b)$ is positive-definite then the system (11) hasn't got any limit cycles.*

Theorem 5.3 *If system (11) admits an elliptic universal section then it has an infinite number of elliptic universal sections.*

Theorem 5.4 *If system (11) has two distinguished elliptic universal sections, corresponding to the functions*

$$\varphi_i = a_i x_i + b_i y_i, \quad i = \overline{1,2} \quad \textit{and} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

then all limit cycles of system (11) belong to the compact set

$$\Omega = \Omega_{\varphi_1} \cap \Omega_{\varphi_2}. \tag{13}$$

Collorary 5.1 *If system (11) admits an elliptic universal section then all limit cycles of system belong to the compact set.*

Theorem 5.5 *If $\delta(a,b)$ is alternating quadric form then all limit cycles of system (11) belong to the compact set.*

6 Conclusion

The suggested method can be efficiently used for solving the localization problem for periodic orbits of ordinary differential systems in various fields of science and technology. Here we use it for spin generator, the Van der Pole equation and linear-quadric systems.

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