Enhanced brackets

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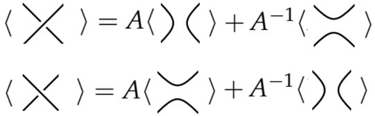
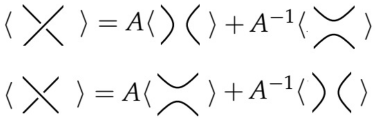
Kauffman Bracket

Definition of Kauffman Bracket (L. Kauffman, 1987):
For an unoriented diagram D, $\langle D \rangle$ is a Laurent polynomial in a single variable A defined by the three following axioms.
1. $\langle \bigcirc \rangle = 1$ where $\bigcirc$ denotes the diagram of unknot with no crossings.
2. Delta: $\langle D \cup \bigcirc \rangle = \delta \langle D \rangle$ where $\delta = -A^{-2} - A^2$.
$\langle D \cup \bigcirc \rangle$ denotes the diagram D together with a single component that does not cross itself or D.
3. Skein relation:



Kauffman Bracket



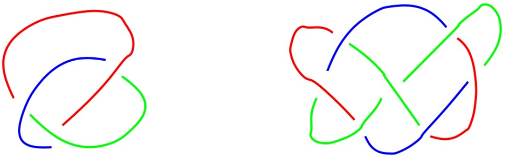
Calculation: The bracket polynomial can be calculated in two ways.

1. Inductively use the skein relation.
2. Simultaneously apply the skein relation to all crossings.

$\langle L \rangle = \sum_S A^{a(s)} A^{-b(S)} (-A^2 - A^{-2})^{|S|-1}$

Approach one

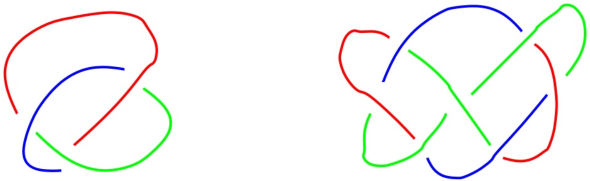
A knot (or a link) is tricolorable if each strand can be colored in one of three colors with the following rules:
At least two colors are used.
At each crossing, either all three colors are present or only one color is present.
If a tricoloring uses only one color we say that it is a trivial tricoloring.
The number of different tricolorings (trivial cloring is allowed) is denoted by $tri(D)$.



Trefoil and 7_4 are tricolorable

Approach one

Lemma [Jozef H. Przytycki 06] $tri(L)$ is always a power of 3.
Theorem [Jozef H. Przytycki 06] (a) $tri(L) = 3|V_L^2(e^{2\pi i/6})|$
(b) $tri(L) = 3|FL(1, -1)|$
$V(7_4) = t - 2 * t^2 + 3 * t^3 - 2 * t^4 + 3 * t^5 - 2 * t^6 + t^7 - t^8$,
$tri(7_4) = 9$.
Hence 7_4 has only one nontrivial coloring up to permutation of the colors.



Trefoil and 7_4 are tricolorable

Approach one

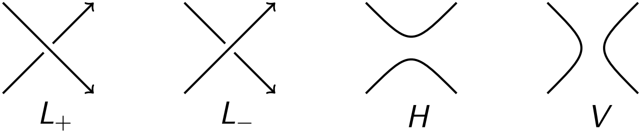


Figure: Local crossings.

Using three colors, if the the three arcs have same color

$< L_+ >= x < H > +x^{-1} < V >$

$< L_- >= x^{-1} < H > +x < V >.$

Otherwise, $< L_+ >= y < H > +y^{-1} < V >$

$< L_- >= y^{-1} < H > +y < V >.$

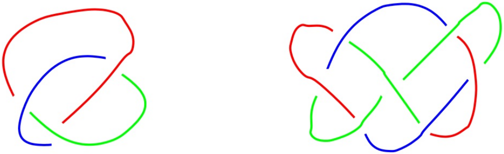
A two variable Jones.

One can easily check that this bracket is invariant under Reidemeister moves II and III. For Reidemeister moves I, the three arcs always have same color. Hence if we let $V(D) = (-x^3)^{-w(D)} < D >$, we shall get a knot invariant.

Theorem

$V(D) = (-x^3)^{-w(D)} < D >$ is a two variable knot invariant.

An Application



Trefoil and 7_4 are tricolorable

7_4 is alternating, hence the bracket is "faithful". Therefore, for any diagram, any nontrivial tricoloring, there exists one crossing with same color, and at least 6 crossings with different colors.

Approach two

S. Nelson, M. Orrison, V. Rivera [2016] introduced the following way to enhance the bracket polynomial.
A link diagram can bicolored as follows. Choose two colors, say, solid and dotted. The crossing points divide any link component into even number of segments. Pick one segment and assign one color to it. Then change to another color whenever pass one crossing point. Do this to each component, we get a bicolor link diagram.

Another Bracket

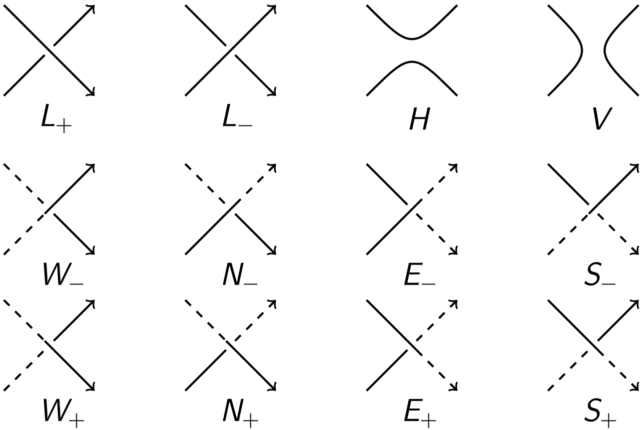


Figure: Local crossings.

S. Nelson, M. Orrison and V. Rivera's construction

$$
\begin{aligned}
N_+ &= H + tV, & N_- &= H + (1+t^2)V, \\
S_+ &= H + tV, & S_- &= H + (1+t^2)V, \\
E_+ &= (1+t)H + (t+t^2)V, & E_- &= tH + V, \\
W_+ &= (1+t^2)H + V, & W_- &= (t+t^2)H + (1+t)V, \\
& < D \sqcup \bigcirc > = (1+t+t^2) < D > .
\end{aligned}
$$

S. Nelson, M. Orrison and V. Rivera's construction

$$
\begin{aligned}
N_+ &= H + tV, & N_- &= H + (1+t^2)V, \\
S_+ &= H + tV, & S_- &= H + (1+t^2)V, \\
E_+ &= (1+t)H + (t+t^2)V, & E_- &= tH + V, \\
W_+ &= (1+t^2)H + V, & W_- &= (t+t^2)H + (1+t)V,
\end{aligned}
$$

$$
< D \sqcup \bigcirc > = (1+t+t^2) < D > .
$$

- Their invariant Φ_X^β takes value in $Z_2[t]/(1+t+t^3)$.

S. Nelson, M. Orrison and V. Rivera's construction

$$\begin{aligned} N_+ &= H + tV, & N_- &= H + (1+t^2)V, \\ S_+ &= H + tV, & S_- &= H + (1+t^2)V, \\ E_+ &= (1+t)H + (t+t^2)V, & E_- &= tH + V, \\ W_+ &= (1+t^2)H + V, & W_- &= (t+t^2)H + (1+t)V, \end{aligned}$$
$$< D \sqcup \bigcirc > = (1+t+t^2) < D > .$$

- $$\begin{cases} N_+ = a_n H + b_n V, & N_- = a'_n H + b'_n V, \\ S_+ = a_s H + b_s V, & S_- = a'_s H + b'_s V, \\ E_+ = a_e H + b_e V, & E_- = a'_e H + b'_e V, \\ W_+ = a_w H + b_w V, & W_- = a'_w H + b'_w V. \\ < D \sqcup \bigcirc >= d < D > . \end{cases} \tag{1}$$

Reidemeister move II

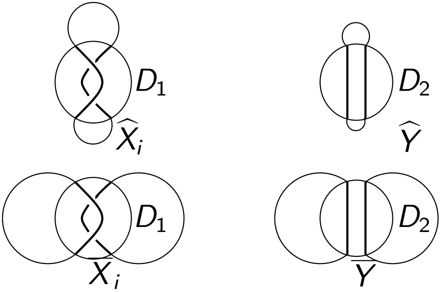


Figure: Outside smoothing patterns

So we have

$$< L_1 >= f_1 < \overline{X}_i > + f_2 < \widehat{X}_i >$$

$$< L_2 >= f_1 < \overline{Y} > + f_2 < \widehat{Y} >$$

Reidemeister move II

$(a_w a'_e)d + (a_w b'_e + b_w a'_e + b_w b'_e d) = d$ and
$(a_w a'_e) + (a_w b'_e + b_w a'_e + b_w b'_e d)d = 1.$

Reidemeister move II

$(a_w a'_e)d + (a_w b'_e + b_w a'_e + b_w b'_e d) = d$ and
$(a_w a'_e) + (a_w b'_e + b_w a'_e + b_w b'_e d)d = 1.$

- Let $x = a_w a'_e, y = a_w b'_e + b_w a'_e + b_w b'_e d$. Then for the linear system of equations $xd + y = d, x + yd = 1$, one can get solutions

Reidemeister move II

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- Let $x = a_w a'_e, y = a_w b'_e + b_w a'_e + b_w b'_e d$. Then for the linear system of equations $xd + y = d, x + yd = 1$, one can get solutions
- $\{x = 1, y = 0\}, \{d = 1, x + y = 1\}, \{d = -1, x - y = 1\}$.

Reidemeister move II

$(a_w a'_e)d + (a_w b'_e + b_w a'_e + b_w b'_e d) = d$ and
$(a_w a'_e) + (a_w b'_e + b_w a'_e + b_w b'_e d)d = 1.$

- Let $x = a_w a'_e, y = a_w b'_e + b_w a'_e + b_w b'_e d$. Then for the linear system of equations $xd + y = d, x + yd = 1$, one can get solutions
- $\{x = 1, y = 0\}, \{d = 1, x + y = 1\}, \{d = -1, x - y = 1\}$.
- For the solution $\{x = 1, y = 0\}$, we have $a_w a'_e = 1, a_w b'_e + b_w a'_e + b_w b'_e d = 0$.

Reidemeister move II

$(a_w a_e')d + (a_w b_e' + b_w a_e' + b_w b_e' d) = d$ and
$(a_w a_e') + (a_w b_e' + b_w a_e' + b_w b_e' d)d = 1.$

- Let $x = a_w a_e', y = a_w b_e' + b_w a_e' + b_w b_e' d$. Then for the linear system of equations $xd + y = d, x + yd = 1$, one can get solutions
- $\{x = 1, y = 0\}, \{d = 1, x + y = 1\}, \{d = -1, x - y = 1\}$.
- For the solution $\{x = 1, y = 0\}$, we have $a_w a_e' = 1, a_w b_e' + b_w a_e' + b_w b_e' d = 0$.
- Then $a_e' = \frac{1}{a_w}, d = -(\frac{a_w}{b_w} + \frac{b_w}{a_w})$.

Reidemeister move II

$(a_w a'_e)d + (a_w b'_e + b_w a'_e + b_w b'_e d) = d$ and
$(a_w a'_e) + (a_w b'_e + b_w a'_e + b_w b'_e d)d = 1.$
There are other solutions

$$\{d = 1, x + y = 1\}, \{d = -1, x - y = 1\}.$$

For simplicity, we first consider the case $\{x = 1, y = 0\}$ here.

Oriented Reidemeister moves

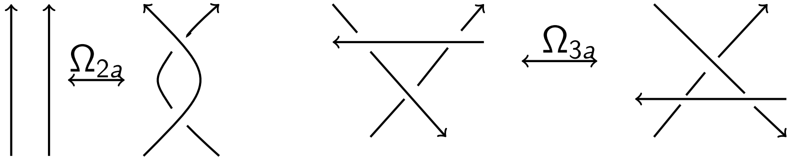


Figure: Reidemeister move two and three.

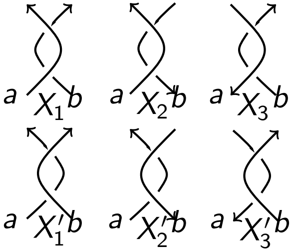


Figure: Oriented Reidemeister move II

All equations from Reidemeister move II

$$\begin{cases} a_w a'_e = 1, a_w b'_e + b_w a'_e + b_w b'_e d = 0 \\ a_e a'_w = 1, a_e b'_w + b_e a'_w + b_e b'_w d = 0 \\ a_s a'_s = 1, a_s b'_s + b_s a'_s + b_s b'_s d = 0 \\ a_n a'_n = 1, a_n b'_n + b_n a'_n + b_n b'_n d = 0 \\ b_w b'_e = 1, b_w a'_e + a_w b'_e + a_w a'_e d = 0 \\ b_e b'_w = 1, b_e a'_w + a_e b'_w + a_e a'_w d = 0 \\ b_n b'_n = 1, b_n a'_n + a_n b'_n + a_n a'_n d = 0 \\ b_s b'_s = 1, b_s a'_s + a_s b'_s + a_s a'_s d = 0. \end{cases} \tag{2}$$

Reidemeister move III

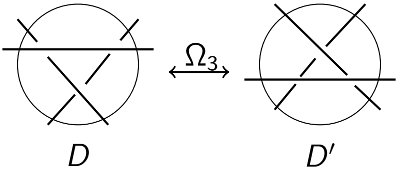


Figure: Reidemeister move three.

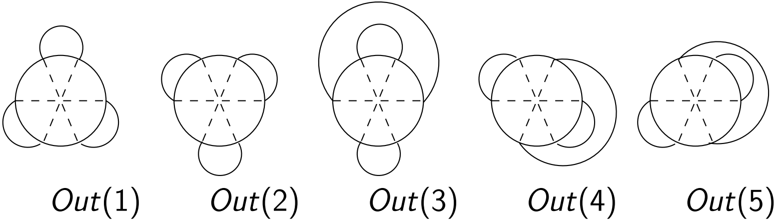


Figure: States outside the disks.

Reidemeister move III

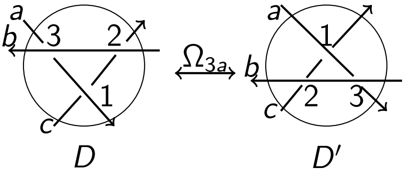


Figure: Reidemeister move Ω_{3a}.

Table: Smooth inside disk.

	$a_1 a_2 a_3$	$a_1 a_2 b_3$	$a_1 b_2 a_3$	$a_1 b_2 b_3$	$b_1 a_2 a_3$	$b_1 a_2 b_3$	$b_1 b_2 a_3$	$b_1 b_2 b_3$
L	B 	A 	A 	E 	A 	F 	G 	C 
L'	D 	C 	C 	E 	C 	F 	G 	A 

Reidemeister move III

Table: Number of components of smoothings of D_i and D_i'

D_i'/D_i	$a_1a_2a_3$	$a_1a_2b_3$	$a_1b_2a_3$	$a_1b_2b_3$	$b_1a_2a_3$	$b_1a_2b_3$	$b_1b_2a_3$	$b_1b_2b_3$
$Out(1)$	4/2	3/1	3/1	2/2	3/1	2/2	2/2	1/3
$Out(2)$	2/4	1/3	1/3	2/2	1/3	2/2	2/2	3/1
$Out(3)$	3/3	2/2	2/2	3/3	2/2	1/1	1/1	2/2
$Out(4)$	3/3	2/2	2/2	1/1	2/2	1/1	3/3	2/2
$Out(5)$	3/3	2/2	2/2	1/1	2/2	3/3	1/1	2/2

To get $< D_i >=< D_i' >$, the second row of table gives the following equation.
$a_na_n'a_nd^4 + a_na_n'b_nd^3 + a_nb_n'a_nd^3 + a_nb_n'b_nd^2 + b_na_n'a_nd^3 + b_na_n'b_nd^2 + b_nb_n'a_nd^2 + b_nb_n'b_nd = a_sa_s'a_sd^2 + a_sa_s'b_sd + a_sb_s'a_sd + a_sb_s'b_sd^2 + b_sa_s'a_sd + b_sa_s'b_sd^2 + b_sb_s'a_sd^2 + b_sb_s'b_sd^3$.
The above is the equation from $Out(1)$. Denote $x_1 = a_na_n'a_n, x_2 = a_na_n'b_n, \cdots, x_8 = b_nb_n'b_n, y_1 = a_sa_s'a_s, \cdots, y_8 = b_sb_s'b_s$, we get a linear equation for variables $x_1, \cdots, y_8$.

Reidemeister move III

Table: Number of components of smoothings of D_i and D_i'

D_i'/D_i	$a_1a_2a_3$	$a_1a_2b_3$	$a_1b_2a_3$	$a_1b_2b_3$	$b_1a_2a_3$	$b_1a_2b_3$	$b_1b_2a_3$	$b_1b_2b_3$
$Out(1)$	4/2	3/1	3/1	2/2	3/1	2/2	2/2	1/3
$Out(2)$	2/4	1/3	1/3	2/2	1/3	2/2	2/2	3/1
$Out(3)$	3/3	2/2	2/2	3/3	2/2	1/1	1/1	2/2
$Out(4)$	3/3	2/2	2/2	1/1	2/2	1/1	3/3	2/2
$Out(5)$	3/3	2/2	2/2	1/1	2/2	3/3	1/1	2/2

To get $< D_i >=< D_i' >$, the second row of table $Out(1)$. gives the following equation.

$$x_1d^4 + x_2d^3 + x_3d^3 + x_4d^2 + x_5d^3 + x_6d^2 + x_7d^2 + x_8d = y_1d^2 + y_2d + y_3d + y_4d^2 + y_5d + y_6d^2 + y_7d^2 + y_8d^3$$

Reidemeister move III

$$\begin{cases} b_n b'_n a_n = b_s b'_s a_s, \quad b_n a'_n b_n = b_s a'_s b_s, \quad a_n b'_n b_n = a_s b'_s b_s \\ b_n b'_n b_n = d a_s a'_s a_s + a_s a'_s b_s + a_s b'_s a_s + b_s a'_s a_s \\ b_s b'_s b_s = d a_n a'_n a_n + a_n a'_n b_n + a_n b'_n a_n + b_n a'_n a_n \end{cases} \tag{3}$$

Table: Number of components of smoothings of D_i and D'_i

	$a1b1c1$	$a1b1c2$	$a1b2c1$	$a1b2c2$	$a2b1c1$	$a2b1c2$	$a2b2c1$	$a2b2c2$
L	$N_+N_-N_+$	$W_+E_-N_+$	$N_+W_-E_+$	$W_+S_-E_+$	$E_+N_-W_+$	$S_+E_-W_+$	$E_+W_-S_+$	$S_+S_-S_+$
L'	$S_+S_-S_+$	$E_+W_-S_+$	$S_+E_-W_+$	$E_+N_-W_+$	$W_+S_-E_+$	$N_+W_-E_+$	$W_+E_-N_+$	$N_+N_-N_+$

Reidemeister move III

$$\begin{cases} b_n b'_n a_n = b_s b'_s a_s, \quad b_n a'_n b_n = b_s a'_s b_s, \quad a_n b'_n b_n = a_s b'_s b_s \\ b_n b'_n b_n = d a_s a'_s a_s + a_s a'_s b_s + a_s b'_s a_s + b_s a'_s a_s \\ b_s b'_s b_s = d a_n a'_n a_n + a_n a'_n b_n + a_n b'_n a_n + b_n a'_n a_n \\ b_n b'_w a_e = b_s b'_e a_w, \quad b_n a'_w b_e = b_s a'_e b_w, \quad a_n b'_w b_e = a_s b'_e b_w \\ b_n b'_w b_e = d a_s a'_e a_w + a_s a'_e b_w + a_s b'_e a_w + b_s a'_e a_w \\ b_s b'_e b_w = d a_n a'_w a_e + a_n a'_w b_e + a_n b'_w a_e + b_n a'_w a_e \\ b_w b'_s a_e = b_e b'_n a_w, \quad b_w a'_s b_e = b_e a'_n b_w, \quad a_w b'_s b_e = a_e b'_n b_w \\ b_w b'_s b_e = d a_e a'_n a_w + a_e a'_n b_w + a_e b'_n a_w + b_e a'_n a_w \\ b_e b'_n b_w = d a_w a'_s a_e + a_w a'_s b_e + a_w b'_s a_e + b_w a'_s a_e \\ b_w b'_e a_n = b_e b'_w a_s, \quad b_w a'_e b_n = b_e a'_w b_s, \quad a_w b'_e b_n = a_e b'_w b_s \\ b_w b'_e b_n = d a_e a'_w a_s + a_e a'_w b_s + a_e b'_w a_s + b_e a'_w a_s \\ b_e b'_w b_s = d a_w a'_e a_n + a_w a'_e b_n + a_w b'_e a_n + b_w a'_e a_n \end{cases}$$

Solution

$$\begin{cases} a_n = a_s = na, b_n = b_s = nb, a_w = wa, b_w = wb, a_e = ea, b_e = eb, \\ a'_n = a'_s = \frac{1}{na}, b'_n = b'_s = \frac{1}{nb}, a'_w = \frac{1}{ea}, b'_w = \frac{1}{eb}, a'_e = \frac{1}{wa}, b'_e = \frac{1}{wb} \\ d = -\frac{a}{b} - \frac{b}{a} \end{cases} \quad (4)$$

An invariant

Let $W(D)$ denote the writhe of the diagram D. Like the construction of Kauffman bracket, let $F(D) = (-\frac{b}{na^2})^{W(D)} < D >$. Then $F(D)$ is invariant under Reidemeister moves. However, it depend on coloring of the link diagram. For a knot diagram, there are only two different colorings. If we change the coloring, we shall get $\overline{F}(D)$. It can be obtained from $F(D)$ as follows. Define $\overline{e} = w, \overline{w} = e$. Then we have

$$\begin{cases} \overline{a_n} = \overline{a_s} = na, \overline{b_n} = \overline{b_s} = nb, \overline{a_w} = ea, \overline{b_w} = eb, \overline{a_e} = wa, \overline{b_e} = wb, \\ \overline{a_n}' = \overline{a_s}' = \frac{1}{na}, \overline{b_n}' = \overline{b_s}' = \frac{1}{nb}, \overline{a_w}' = \frac{1}{wa}, \overline{b_w}' = \frac{1}{wb}, \overline{a_e}' = \frac{1}{ea}, \overline{b_e}' = \frac{1}{eb} \\ \overline{d} = -\frac{a}{b} - \frac{b}{a} \end{cases} (5).$$

An invariant

In general, For a link L, choose one link diagram D, let Λ be the set of all colorings. If one choose one coloring $\lambda \in \Lambda$, one will get $F(D, \lambda)$. Now we have the following theorem.

Theorem

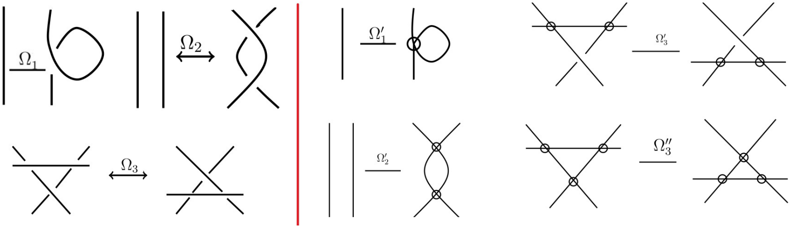
Using the skein relations (1), if the variables satisfies equation (7), then $\{F(D), \overline{F}(D)\}$ *is a knot invariant.*
For a link L, choose one link diagram D, then $F(L) = \{F(D, \lambda) \mid \lambda \in \Lambda\}$ *is a multiple-valued link invariant.*

Remark

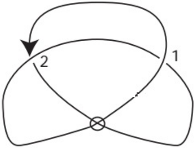
Approach one and two do not give new results for classical knots.
They do give stronger invariants for virtual knots.

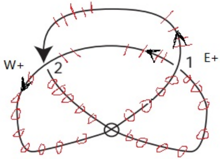
Virtual links (L. Kauffman, 1999)

A virtual link diagram is a planar 4-valent graph has three type of crossing types: overcrossing, undercrossing or virtual crossing.
Two virtual link diagrams are equivalent if there exists a sequence of usual and generalised Reidemeister moves, transforming one diagram to the other one.

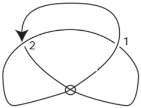


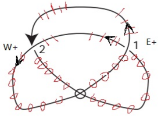
The following virtual knot has E_+, W_+ type crossings.

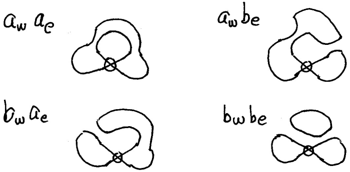




An example.







Its bracket $= a_w a_e + a_w b_e + b_w a_e + b_w b_e d =$
$waea + waeb + wbea + wbeb(-\frac{a}{b} - \frac{b}{a}) = we(a^2 + ab - \frac{b^3}{a})$
Hence it is not classical.

An application

$$\begin{cases} a_n = a_s = 1, b_n = b_s = t, a_w = 1 + t^2, b_w = 1, a_e = 1 + t, b_e = t + t^2, \\ a'_n = a'_s = 1, b'_n = b'_s = 1 + t^2, a'_w = t + t^2, b'_w = 1 + t, a'_e = t, b'_e = 1 \\ d = 1 + t + t^2 \end{cases} \quad (6)$$

The coefficients lie in $Z_2[t]/(1 + t + t^3)$. We can lift them to $Z[t, t^{-1}]$ as follows.

$$\begin{cases} a_n = a_s = 1, b_n = b_s = t, a_w = 1 + t^2, b_w = t(1 + t^2), a_e = 1 + t, b_e = t(1 + t), \\ a'_n = a'_s = 1, b'_n = b'_s = 1/t, a'_w = \dfrac{1}{1+t}, b'_w = \dfrac{1}{t(1+t)}, a'_e = \dfrac{1}{1+t^2}, b'_e = \dfrac{1}{t(1+t^2)} \\ d = -t - \dfrac{1}{t} \end{cases} \quad (7)$$

Approach three

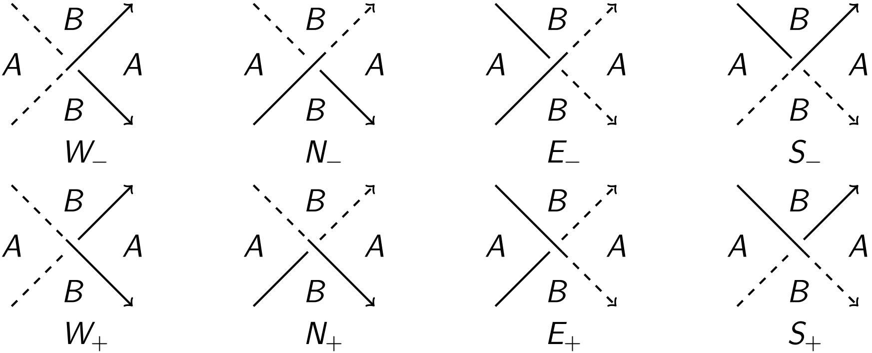


Figure: Coloring arcs and regions.

Skein relations

$$\begin{cases} N_+ = a_n H + b_n V, & N_- = a'_n H + b'_n V, \\ S_+ = a_s H + b_s V, & S_- = a'_s H + b'_s V, \\ E_+ = a_e H + b_e V, & E_- = a'_e H + b'_e V, \\ W_+ = a_w H + b_w V, & W_- = a'_w H + b'_w V. \\ < D \sqcup \bullet >= d < D > . \end{cases} \tag{8}$$

$$\begin{cases} N_+ = \overline{a}_n H + \overline{b}_n V, & N_- = \overline{a}'_n H + \overline{b}'_n V, \\ S_+ = \overline{a}_s H + \overline{b}_s V, & S_- = \overline{a}'_s H + \overline{b}'_s V, \\ E_+ = \overline{a}_e H + \overline{b}_e V, & E_- = \overline{a}'_e H + \overline{b}'_e V, \\ W_+ = \overline{a}_w H + \overline{b}_w V, & W_- = \overline{a}'_w H + \overline{b}'_w V. \\ < D \sqcup \bigcirc >= \overline{d} < D > . \end{cases} \tag{9}$$

For Knots

$$
\begin{cases}
a_s a'_s = 1, a_s b'_s + b_s a'_s + b_s b'_s d = 0 \\
a_n a'_n = 1, a_n b'_n + b_n a'_n + b_n b'_n d = 0 \\
b_n b'_n = 1, b_n a'_n + a_n b'_n + a_n a'_n d = 0 \\
b_s b'_s = 1, b_s a'_s + a_s b'_s + a_s a'_s d = 0 \\
\overline{b}_n \overline{b}'_n \overline{a}_n = b_s b'_s a_s, \qquad \overline{b}_n \overline{a}'_n \overline{b}_n = b_s a'_s b_s, \qquad \overline{a}_n \overline{b}'_n \overline{b}_n = a_s b'_s b_s \\
\overline{b}_n \overline{b}'_n \overline{b}_n = d a_s a'_s a_s + a_s a'_s b_s + a_s b'_s a_s + b_s a'_s a_s \\
\overline{b}_s \overline{b}'_s \overline{b}_s = d a_n a'_n a_n + a_n a'_n b_n + a_n b'_n a_n + b_n a'_n a_n \\
\overline{a}_s \overline{a}'_s = 1, \overline{a}_s \overline{b}'_s + \overline{b}_s \overline{a}'_s + \overline{b}_s \overline{b}'_s d = 0 \\
\overline{a}_n \overline{a}'_n = 1, \overline{a}_n \overline{b}'_n + \overline{b}_n \overline{a}'_n + \overline{b}_n \overline{b}'_n d = 0 \\
\overline{b}_n \overline{b}'_n = 1, \overline{b}_n \overline{a}'_n + \overline{a}_n \overline{b}'_n + \overline{a}_n \overline{a}'_n d = 0 \\
\overline{b}_s \overline{b}'_s = 1, \overline{b}_s \overline{a}'_s + \overline{a}_s \overline{b}'_s + \overline{a}_s \overline{a}'_s d = 0 \\
b_n b'_n a_n = \overline{b}_s \overline{b}'_s \overline{a}_s, \qquad b_n a'_n b_n = \overline{b}_s \overline{a}'_s \overline{b}_s, \qquad a_n b'_n b_n = \overline{a}_s \overline{b}'_s \overline{b}_s \\
b_n b'_n b_n = d \overline{a}_s \overline{a}'_s \overline{a}_s + \overline{a}_s \overline{a}'_s \overline{b}_s + \overline{a}_s \overline{b}'_s \overline{a}_s + \overline{b}_s \overline{a}'_s \overline{a}_s \\
b_s b'_s b_s = d \overline{a}_n \overline{a}'_n \overline{a}_n + \overline{a}_n \overline{a}'_n \overline{b}_n + \overline{a}_n \overline{b}'_n \overline{a}_n + \overline{b}_n \overline{a}'_n \overline{a}_n
\end{cases}
\tag{10}
$$

Solution

$$\begin{cases} a_n = na, a_s = sa, b_n = nb, b_s = nb, \\ a'_n = \dfrac{1}{na}, a'_s = \dfrac{1}{sa}, b'_n = \dfrac{1}{nb}, b'_s = \dfrac{1}{sb} \\ d = -\dfrac{a}{b} - \dfrac{b}{a}, \overline{d} = -\dfrac{\overline{a}}{\overline{b}} - \dfrac{\overline{b}}{\overline{a}} \\ \overline{a}_n = \overline{na}, \overline{a}_s = \overline{sa}, \overline{b}_n = \overline{nb}, \overline{b}_s = \overline{nb}, \\ \overline{a}'_n = \dfrac{1}{\overline{na}}, \overline{a}'_s = \dfrac{1}{\overline{sa}}, \overline{b}'_n = \dfrac{1}{\overline{nb}}, \overline{b}'_s = \dfrac{1}{\overline{sb}} \end{cases} \tag{11}$$

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Thank you for attention!