Homotopy Aspects of Braids and Links

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4th China-Russia Workshop on Knot Theory and Related Topics, Moscow, 3-8 July 2017

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Braids and double loop spaces

Configuration space
$F(M,n)=\{(z_1,\ldots,z_n)\in M^{\times n}\mid z_i\neq z_j \text{ for } i\neq j\}$.

$F(\mathbb{R}^2,n)\simeq K(P_n,1)$ and $F(\mathbb{R}^2,n)/\Sigma_n\simeq K(B_n,1)$, where B_n is the n-strand Artin braid group and P_n is the pure braid group.

Configuration space with labels in a space X
$C(M;X)=\bigcup_n F(M,n)\times_{\Sigma_n} X^n/\sim$, where
$(z_1,\ldots,z_n;x_1,\ldots,x_n)\sim(z_1,\ldots,z_{n-1};x_1,\ldots,x_{n-1})$ if $x_n=*$.

(Segal'73, first by May'72 (LNM Vol. 271), ideas earlier by Boardman-Vogt'68): $C(\mathbb{R}^k;X)\simeq\Omega^k\Sigma^k X$ if X path-connected.

Quillen's plus construction on braid groups

A consequence of Segal's work: There is a map $K(B_\infty, 1) \to \Omega^2_0 S^2$ inducing an isomorphism on homology, where $\Omega^2_0 S^2$ is the path-connected component of $\Omega^2 S^2$ containing the base-point.

Cohomology of braid groups B_n was first studied by **Arnold'70**, also studied by **D. B. Fuks'70** and **F. R. Cohen'73**.

Quillen's plus construction: $q_X : X \to X^+$ is a pointed cofibration which induces isomorphisms on homology with abelian local coefficients and epimorphism on π_1 with $\mathrm{Ker}(\pi_1(q_X))$ the maximal perfect subgroup of $\pi_1(X)$.

$K(B_\infty, 1)^+ \simeq \Omega^2_0 S^2$.

Brunnian braids

A **Brunnian braid** (called **smooth braids** by Makanin) means a braid that becomes trivial after removing any one of its strands.

- **Levinson'75**: (called **decomposable braids**) Let

  $$t_i = \sigma_i \sigma_{i+1} \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2}^{-1} \cdots \sigma_i^{-1}$$

  for $1 \leq i \leq n-1$. Let $R_i = \langle\langle t_i \rangle\rangle$ be the normal closure of t_i in the pure braid group P_n. Then

  $$\mathrm{Brun}_4(D^2) = [[R_1, R_2], R_3] \cdot [[R_1, R_3], R_2].$$

- **Generalized by Jingyan Li- Wu'11**: For each $n \geq 3$,

  $$\mathrm{Brun}_n(D^2) = \prod_{\sigma \in \Sigma_{n-1}} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n-1)}].$$

- **Makanin's Question'80** on determining generators for Brunnian braids. Answered by **Gurzo' 81**, and **Johnson'82**.

Brunnian braids

- **Berrick-Cohen-Wong-Wu'06:** There is an exact sequence

$$1 \to \mathrm{Brun}_{n+1}(S^2) \to \mathrm{Brun}_n(D^2) \to \mathrm{Brun}_n(S^2) \to \pi_{n-1}(S^2) \to 1$$

  for $n \geq 5$.

- **Badakov-Mikhailov-Vershinin-Wu'12:** Let M be a connected 2-manifold and let $n \geq 2$. The inclusion $f\colon D^2 \hookrightarrow M$ induces a group homomorphism

$$f_*\colon P_n(D^2) \longrightarrow P_n(M).$$

  Let $A_{i,j}[M] = f_*(A_{i,j})$ and let $\langle\langle A_{i,j}[M]\rangle\rangle^P$ be the normal closure of $A_{i,j}[M]$ in $P_n(M)$. Let

$$R_n(M) = [\langle\langle A_{1,n}[M]\rangle\rangle^P, \langle\langle A_{2,n}[M]\rangle\rangle^P, \ldots, \langle\langle A_{n-1,n}[M]\rangle\rangle^P]_S$$

Badakov-Mikhailov-Vershinin-Wu'12:

- **Theorem 1.**
  1. If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, then
  $$\mathrm{Brun}_n(M) = R_n(M).$$
  2. If $M = S^2$ and $n \geq 5$, then there is a short exact sequence
  $$R_n(S^2) \hookrightarrow \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2).$$
  3. If $M = \mathbb{R}\mathrm{P}^2$ and $n \geq 4$ then there is a short exact sequence
  $$R_n(\mathbb{R}\mathrm{P}^2) \hookrightarrow \mathrm{Brun}_n(\mathbb{R}\mathrm{P}^2) \twoheadrightarrow \pi_{n-1}(S^2).$$
- **Theorem 2.** The factor groups $P_n(M)/\mathrm{Brun}_n(M)$ and $B_n(M)/\mathrm{Brun}_n(M)$ are finitely presented for each $n \geq 3$.
- **Question 23 in Birman's book'75** on braid, links and mapping class groups: Determine a free basis for $\mathrm{Brun}_n(S^2)$. **Unsolved.** Seems hard question.

$\pi_*(S^k)$ for $k \geq 3$—Mikhailov-Wu

- We give a combinatorial description of $\pi_*(S^k)$ for any $k \geq 3$ by using the free product with amalgamation of pure braid groups.

- Given $k \geq 3,\ n \geq 2$, let P_n be the n-strand Artin pure braid group with the standard generators $A_{i,j}$ for $1 \leq i < j \leq n$. We construct certain (free) explicit subgroup $Q_{n,k}$ of P_n (depending on n and k).

$\pi_*(S^k)$ for $k \geq 3$—Mikhailov-Wu

Now consider the free product with amalgamation

$$P_n *_{Q_{n,k}} P_n.$$

Namely this amalgamation is obtained by identifying the elements y_j in two copies of P_n. Let $A_{i,j}$ be the generators for the first copy of P_n and let $A'_{i,j}$ denote the generators $A_{i,j}$ for the second copy of P_n. Let $R_{i,j} = \langle A_{i,j}, A'_{i,j} \rangle^{P_n *_{Q_{n,k}} P_n}$ be the normal closure of $A_{i,j}, A'_{i,j}$ in $P_n *_{Q_{n,k}} P_n$. Let

$$[R_{i,j} \mid 1 \leq i < j \leq n]_S = \prod_{\{1,2,\ldots,n\}=\{i_1,j_1,\ldots,i_t,j_t\}} [[R_{i_1,j_1}, R_{i_2,j_2}], \ldots, R_{i_t,j_t}]$$

be the product of all commutator subgroups such that each integer $1 \leq j \leq n$ appears as one of indices at least once.

Mikhailov-Wu'13

Theorem. Let $k \geq 3$. The homotopy group $\pi_n(S^k)$ is isomorphic to the center of the group

$$(P_n *_{Q_{n,k}} P_n)/[R_{i,j} \mid 1 \leq i < j \leq n]_S$$

for any n if $k > 3$ and any $n \neq 3$ if $k = 3$.

- **Note.** The only exceptional case is $k = 3$ and $n = 3$. In this case, $\pi_3(S^3) = \mathbb{Z}$ while the center of the group is $\mathbb{Z}^{\oplus 4}$.

Lie algebras of groups

We recall that for a group G the descending central series

$$G = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_i \geq \Gamma_{i+1} \geq \ldots .$$

is defined by the formulae

$$\Gamma_1 = G, \;\; \Gamma_{i+1} = [\Gamma_i, G].$$

The descending central series of a discrete group G gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $L(G)$

$$L_i(G) = \Gamma_i(G)/\Gamma_{i+1}(G).$$

Yang-Baxter Lie algebra

Let $G = P_n$.
Kohno'85: The Lie algebra $L(P_n)$ is the quotient of the free Lie algebra $L[A_{i,j} \mid 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given by the following three relations:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \emptyset, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i < j < k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i < j < k. \end{cases} \tag{1}$$

Braid Commutators and Vassiliev Invariants

Ted Stanford'96: Let L and L' be two links which differ by a braid $p \in \Gamma_n(P_k)$. Let v be a link invariant of order less than n. Then $v(L) = v(L')$.

Here the meaning for two links to differ by a braid p is as follows: Let $\hat{x}$ denote the closure of a braid x. Let b and p be any two braids with the same number of strands. Then $\hat{b}$ and $\hat{pb}$ differ by p.

In brief, **lower central series of pure braid groups $\Longrightarrow$ Vassiliev Invariants**.

Vassiliev Invariants on subgroups of pure braid groups

Let $G \leq P_k$ be a subgroup of P_k. The elements in G give a set of special type of braids. Then the set $\hat{G} = \{\hat{x} \mid x \in G\}$ gives a subset of special type of links.

For detecting the Vassiliev invariants on the special type of links given by $\hat{G}$, a natural way is to consider

$$G = \Gamma_1(P_k) \cap G \geq \Gamma_2(P_k) \cap G \geq \cdots \geq \Gamma_i(P_k) \cap G \geq \Gamma_{i+1}(P_k) \cap G \geq \ldots .$$

The resulting (relative) Lie algebra $L^{P_k}(G) = \bigoplus_{i=1}^{\infty} (\Gamma_i(P_k) \cap G)/(\Gamma_{i+1}(P_k) \cap G)$ is a sub Lie algebra of the Yang-Baxter Lie algebra $L(P_k)$.

Symmetric bracket sum of Lie ideals

Let L be a Lie algebra and $I_1, \ldots, I_n$ ideals of L. The **symmetric bracket sum** of these ideals is defined as

$$[[I_1, I_2], \ldots, I_l]_S := \sum_{\sigma \in \Sigma_l} [[I_{\sigma(1)}, I_{\sigma(2)}], \ldots, I_{\sigma(n)}],$$

where Σ_n is the symmetric group on n letters.

Jingyan Li-Vershinin-Wu'15

Let us denote the ideal

$$L[A_{k,n}, [\cdots[A_{k,n}, A_{j_1,n}], \ldots, A_{j_m,n}] \mid j_i \neq k, n;\ j_i \leq n{-}1, i \leq m;\ m \geq 1]$$

by I_k.

Theorem.
The Lie subalgebra $L^{P_n}(\mathrm{Brun}_n)$ and the symmetric bracket sum $[[I_1, I_2], \ldots, I_{n-1}]_S$ are equal as subalgebras in $L(P_n)$:

$$L^{P_n}(\mathrm{Brun}_n) = [[I_1, I_2], \ldots, I_{n-1}]_S.$$

Brunnian Lie algebra over S^2—Li-Vershinin-Wu, working progress

Let $\mathrm{BrunL}(S^2)_n = \bigcap_{i=1}^{n} \ker(d_i : L(P_n(S^2)) \to L(P_{n-1}(S^2)))$. Let J_i be the image of I_i under the projection $L(P_n) \to L(P_n(S^2))$.

Theorem. There is a short exact sequence

$$[[J_1, J_2], \ldots, J_{n-1}]_S \hookrightarrow \mathrm{BrunL}(S^2)_n \twoheadrightarrow \Lambda_{n-1}(S^2)$$

for $n \geq 5$, where $\Lambda(S^2)$ is the Λ-algebra. Moreover

$$[[J_1, J_2], \ldots, J_{n-1}]_S \leq L^P(\mathrm{Brun}_n(S^2)) \leq \mathrm{BrunL}(S^2)_n$$

with $|L^P(\mathrm{Brun}_n(S^2))/[[J_1, J_2], \ldots, J_{n-1}]_S| = |\pi_{n-1}(S^2)|$ for $n \geq 5$.

Bardakov-Vershinin-Wu'14

Let M be a general connected surface, possibly with boundary components. Consider the braid group $B_n(M)$. Let $d_i\colon B_n(M) \to B_{n-1}(M)$ be the function given by removing the i-th strand for $1 \leq i \leq n$.

Our question. Given an $(n-1)$-stand braid α, does there exist an n-strand braid β such that it is a solution of the system of equations

$$\begin{cases} d_1\beta & = & \alpha \\ \cdots & & \\ d_n\beta & = & \alpha \end{cases}$$

Theorem. Let $M \neq S^2, \mathbb{RP}^2$. Let $\alpha \in B_{n-1}(M)$. Then the above system of equations has a solution if and only if $d_1\alpha = d_2\alpha = \cdots = d_{n-1}\alpha$.

Fengchun Lei-Fengling Li-Wu'14

Let (L, X) be a framed link in S^3 with X a vector field defined in a neighborhood of L perpendicular to the tangent field of L. We can obtain a sequence of link $\mathbb{L} = \{L_0, L_1, \ldots, \ldots\}$, where $L_0 = L$ and L_n is a naive n-cabling of L along the vector field X. By taking the fundamental groups of the link complements, we obtain a simplicial group $G(L, X) = \{\pi_1(S^3 \setminus L_n)\}_{n\geq 0}$.
Let

$$L \cong L^{[1]} \sqcup L^{[2]} \sqcup \cdots \sqcup L^{[p]}$$

be the **splitting decomposition** of the framed link L such that $(L^{[i]}, X|L^{[i]})$ is a nontrivial nonsplittable framed link for $1 \leq i \leq k$ and $(L^{[i]}, X|L^{[i]})$ is a trivial framed link for $k+1 \leq i \leq p$. Then

- **Theorem.** the geometric realization $|G(L; X)|$ is homotopy equivalent to $\Omega(\bigvee^k S^3)$.

Simplicial group $G(L, X)$ detects trivial framed knot

Let K be a framed knot with frame X.

- $G(K, X)$ is a knot invariant given by simplicial group. (K, X) is a trivial framed knot if and only if $G(K, X)$ is contractible. (K, X) is a **non-trivial framed knot** if and only if $G(K, X) \simeq \Omega S^3$.

- Recall that $\pi_2(\Omega S^3) = \pi_3(S^3) = \mathbb{Z}$.

$$\pi_2(G(K, X)) = \begin{cases} 0 & \text{if} \quad (K, X) \text{ a trivial framed knot} \\ \mathbb{Z} & \text{if} \quad (K, X) \text{ a non-trivial framed knot} \end{cases}$$

- For different nontrivial framed knots (K, X) and $(K', , X')$, although $G(K, X)$ and $G(K', L')$ has the same homotopy type, they may be given by different link groups. Simplicial group ring $\mathbb{Z}(G(K, X))$ may give new knot invariants.

Fuquan Fang-Fengchun Lei-Wu'15

Let L be an n-link in a 3-manifold M. Let A_i be the normal closure of the i th meridian. Then the symmetric commutator subgroup

$$[[A_1, \ldots, A_n]_S = \prod_{\sigma\in\Sigma_n} [[A_{\sigma(1)}, A_{\sigma(2)}], \ldots, A_{\sigma(n)}]$$

is a (normal) subgroup of the intersection subgroup $A_1 \cap A_2 \cap \cdots \cap A_n$.

- **Theorem.** Let L be any **strongly nonsplittable** n-link in M with $n \geq 2$. Then

  $$\pi_n(M) \cong A_1 \cap A_2 \cap \cdots \cap A_n/[[A_1, \ldots, A_n]_S.$$

  for any $n \geq 2$.
- In particular, if $M = S^3$, this gives a description of homotopy groups of S^3 in terms of link groups.

Question

Let L be an n-link in S^3. A knot K in $S^3 \smallsetminus L$ is called *almost trivial* if K bounds a disk in $S^3 \smallsetminus d_i L$ for each $1 \leq i \leq n$.

A knot K in $S^3 \smallsetminus L$ is called *weakly almost trivial* if K represents an element in $A_1 \cap A_2 \cap \cdots \cap A_n$.

A knot K in $S^3 \smallsetminus L$ is called *commutatorized* if K represents an element in the symmetric commutator subgroup $[[A_1, A_2], \ldots, A_n]_S$.

Question. Let L be an n-link. Let K be a weakly almost trivial knot in the link complement $S^3 \smallsetminus L$. Does there exist a connected sum decomposition

$$K = K' \# K''$$

such that K' is a commutaorized knot in $S^3 \smallsetminus L$ and K'' is an almost trivial knot in $S^3 \smallsetminus L$?

Thank You!