Representation volume and higher dimensional geometries

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Moscow, July 2017

How to measure the complexity of a manifold?

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- *"the least real amount of simplices to build a fundamental cycle"*
- the ℓ^1–norm infimum of a rational singular cycle that represents the fundamental class

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- virtually nonvanishing for graph manifolds, mixed manifolds, and hyperbolic mainifolds of dimension 3 [DLSW 2016]

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- introduced to detect finiteness of the mapping degree set $D(M, N)$ between closed oriented manifolds
- interesting especially when v is always finite and is sometimes nonvanishing

What is the definition of Seifert volume?

Representation volume for Seifert space geometry

- SETTINGS:
  (X, G, ω_X) [the geometry $\left(\widetilde{\mathrm{SL}}_2(\mathbb{R}), \mathrm{Iso}_e\widetilde{\mathrm{SL}}_2(\mathbb{R})\right)$ and a G–invariant volume form ω_X of X]
  $\mathcal{R}(M, G)$ [the set of representations $\rho\colon \pi_1(M) \to G$ for a closed oriented 3-manifold M]
- OUTPUT:
  $\mathrm{Vol}_{G,X,\omega_X}(M, -)\colon \mathcal{R}(\pi_1 M, G) \to \mathbb{R}$ [well defined by the expression $\mathrm{Vol}_{G,X,\omega_X}(M, \rho) = \int_{\mathcal{F}} D_\rho^*\omega_X$ where $D_\rho\colon \widetilde{M} \to X$ is a ρ–equivariant developing map]
  $\mathrm{V}(M, G) \in [0, +\infty)$ [well defined by the expression $\mathrm{V}(M, G) = \sup_{\rho\in\mathcal{R}(\pi_1 M, G)} |\mathrm{Vol}_{G,X,\omega_X}(M, \rho)|$]
- PROPERTIES:
  domination inequality, finiteness, and nontriviality [Brooks–Goldman]

And representation volumes in general?

Representation volume in Seifert space geometry

- SETTINGS:
  (X, G, ω_X)
  $\mathcal{R}(M, G)$
- OUTPUT:
  $\mathrm{Vol}_{G,X,\omega_X}(M, -)\colon \mathcal{R}(\pi_1 M, G) \to \mathbb{R}$
  $\mathrm{V}(M, G) \in [0, +\infty)$
- PROPERTIES:
  domination inequality, finiteness, and nontriviality

Representation volume in general

- SETTINGS:
  (X, G, ω_X) [a connected real Lie group G, and a proper and contractible G–homogeneous space X, and a G–invariant volume form ω_X of X]
  $\mathcal{R}(M, G)$ [requiring $\dim M = \dim X$]
- OUTPUT:
  $\mathrm{vol}_{G,X,\omega_X}(M, -)\colon \mathcal{R}(\pi_1 M, G) \to \mathbb{R}$
  $\mathrm{V}(M, G) \in [0, +\infty]$
- PROPERTIES:
  domination inequality [OK], finiteness(?), and nontriviality(?)

Theorem (DLSW 2017)

Suppose that G is a connected real Lie group which contains a closed cocompact connected semisimple subgroup. Let $X = G/H$ be a homogeneous space furnished with a G–invariant volume form, where H is a maximal compact subgroup of G. Then for any oriented closed smooth manifold M of the same dimension as X, the volume function

$$\mathrm{vol}_G \colon \mathcal{R}(\pi_1(M), G) \to \mathbb{R}$$

takes only finitely many values on the space of representations $\mathcal{R}(\pi_1(M), G)$. Moreover, there exists some aspherical M for which vol_G is not constantly zero.

What does the theorem say concretely?

List by dimension

The pair (X, G) as assumed is a model geometry according to W. P. Thurston. The theorem applies to ...

- 2d geometries: $\left(\mathbb{H}^2, \mathrm{PSL}(2,\mathbb{R})\right)$
- 3d geometries: $\left(\mathbb{H}^3, \mathrm{PSL}(2,\mathbb{C})\right)$, $\left(\widetilde{\mathrm{SL}}_2(\mathbb{R}), \widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}\right)$
- higher dimensional families:
  - symmetric spaces of noncompact type
  - $\left(\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_\alpha \widetilde{\mathrm{SL}}_2(\mathbb{R}), \mathrm{Iso}_e(\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_\alpha \widetilde{\mathrm{SL}}_2(\mathbb{R}))\right)$ for a rational α.

Compare the non-example families:

- symmetric spaces of compact type
- abelian, nilpotent, and solvable geometries
- $\left(\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_\alpha \widetilde{\mathrm{SL}}_2(\mathbb{R}), \mathrm{Iso}_e(\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times_\alpha \widetilde{\mathrm{SL}}_2(\mathbb{R}))\right)$ for an irrational α.

"Happy families are all alike; every unhappy family is unhappy in its own way."

Comparison of volumes

The following comparisons of representation volumes hold in general:

- (the domination inequality) $\mathrm{V}(M', G) \geq |\deg(f)| \cdot \mathrm{V}(M, G)$ if $f\colon M' \to M$
- (the induction inequality) $\mathrm{V}(M, G') \leq \mathrm{V}(M, G)$ if $\phi\colon G' \to G$
- (the connected sum inequality) $\mathrm{V}(\#_i M_i,\, G) \leq \sum_i \mathrm{V}(M_i, G)$ for finite connected sums
- (the product inequality) $\mathrm{V}\left(\prod_i M_i, \prod_i G_i\right) \geq \max_{\sigma \in \mathfrak{S}(I)} \left\{\prod_i \mathrm{V}\left(M_i, G_{\sigma(i)}\right)\right\}$ for finite direct products

None of the above are equalities in general.

Example: strict domination inequality

Take G to be either $\mathrm{PSL}(2,\mathbb{C})$ or $\widetilde{\mathrm{SL}_2}(\mathbb{R})$.

There exists a closed oriented 3-manifold M with vanishing $\mathrm{V}(M,G)$, whereas $\mathrm{V}(M',G) > 0$ holds for some finite cover M' of M.
[DLW 2015]

So $\mathrm{V}(M',G) > |\deg(f)|\,\mathrm{V}(M,G)$ holds for the covering map $f\colon M' \to M$.

Example: strict induction inequality

Take G to be $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ and G' to be $\widetilde{\mathrm{SL}_2}(\mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$.

Consider an oriented closed 3–manifold M which is a Seifert fibered space with the symbol $(2, 0; 3/2)$. The base 2–orbifold is a closed oriented surface of genus 2 and with a cone point of order 2, so it has Euler characteristic $\chi = -5/2$. The (orbifold) Euler number of the Seifert fibration equals $e = 3/2$.

Then an explicit formula shows
$\mathrm{V}(M, G') = 4\pi^2 \times (-5/2)^2/|3/2| = 50\pi^2/3$ whereas
$\mathrm{V}(M, G) = 4\pi^2 \times 1^2 \times (3/2) = 6\pi^2$.

So $V(M, G') < V(M, G)$ holds for this case.

Example: strict connected sum inequality

Take G to be $\mathrm{PSL}(2,\mathbb{C})$ and Γ be a torsion-free uniform lattice of G. Denote by $M = \mathbb{H}^3/\Gamma$ be the closed hyperbolic 3–manifold with the induced orientation.

The set of volumes for all representations of $\pi_1(M) \cong \Gamma$ in G consists of finitely many real values $v_1 < v_2 < \cdots < v_s$. If we require further that M admits no orientation-reversing self-homeomorphism, it is implied by the volume rigidity that $v_s = \mathrm{Vol}_{\mathbb{H}^3}(M)$ and $|v_1| < v_s$ [Besson–Courtois–Gallot]. The set of volumes of the orientation-reversal $-M$ consists of $-v_s < \cdots < -v_2 < -v_1$.

Then the set of volumes for the connected sum $M\#(-M)$ consists of all the values $v_i - v_j$, and $\mathrm{V}(M\#(-M), G) = |v_s - v_1| < 2v_s$. But we have $\mathrm{V}(M, G) = \mathrm{V}(-M, G) = v_s$.

So $\mathrm{V}(M\#(-M), G) < \mathrm{V}(M, G) + \mathrm{V}(-M, G)$ in this case.

Example: strict product inequality

Take G to be $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{C})$. Let S be the unit tangent bundle of a closed oriented hyperbolic surface, and H be a closed oriented hyperbolic 3–manifold. Let $M_S = S \times S$ and $M_H = H \times H$.

It can be shown that $\mathrm{V}(M_S, G)$ must be zero, by computing the cohomology class of the pull-back volume form. However, it is obvious that $\mathrm{V}(M_S \times M_H, G \times G) > 0$.

So $\mathrm{V}(M_S \times M_H, G \times G) > \mathrm{V}(M_S, G) \times \mathrm{V}(M_H, G)$ in this case.

How is the theorem proved?

A reminder of the statement

Theorem (DLSW 2017)

Suppose that G is a connected real Lie group which contains a closed cocompact connected semisimple subgroup. Let $X = G/H$ be a homogeneous space furnished with a G–invariant volume form, where H is a maximal compact subgroup of G. Then for any oriented closed smooth manifold M of the same dimension as X, the volume function

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Ingredients of the proof

- Based on the semisimple case: If G is semisimple, the idea has been sketched by Goldman.
- Working with full central extension: When G has torsion-free (finitely generated abelian) center Z, we prove the theorem for the full central extension $G \times_Z Z_{\mathbb{R}}$ where $Z_{\mathbb{R}} = Z \otimes \mathbb{R}$. We characterize $\mathcal{R}(\pi, G \times_Z Z_{\mathbb{R}})$ and generalize Goldman's idea to the full central extension, which has a reductive Lie algebra.
- Working with cocompactly closed semisimple Lie groups: Any connected real Lie group G that contains a closed cocompact connected semisimple Lie subgroup fits into an exact sequence of homomorphisms of Lie groups

$$\{0\} \longrightarrow Z(G)_{\texttt{tor}} \longrightarrow G \longrightarrow \hat{G}_{\mathbb{R}} \longrightarrow T \longrightarrow \{0\}$$

where $Z(G)_{\texttt{tor}}$ is the maximal compact central subgroup of G, and T is a connected compact abelian Lie group, and $\hat{G}_{\mathbb{R}}$ is the full central extension of a connected semisimple Lie group $\hat{G}$ with torsion-free center.

Conclusions

- It seems appropriate to treat representation volumes as associated with geometries in the sense of Thurston.
- For cocompactly closed semisimple Lie groups, the associated representation volume is essentially determined by its semisimple part.
- We suspect that these essentially cover all the interesting representation volumes.

Thank you for your attention.